Reflection Symmetry in the Game of Daisy

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Symmetry has interesting applications in the world of mathematics—more specifically, the area of mathematics involving game analysis. Let's explore an example of reflection symmetry in the form of the Daisy game.

This game requires two players and a thirteen-petal flower (Fig. 1). The rules are as follows: each player can choose to remove either one or two adjacent petals on his or her turn. Players alternate turns, and the person to remove the last petal wins. Seems simple enough, but the question is: what if you could guarantee a win every time?

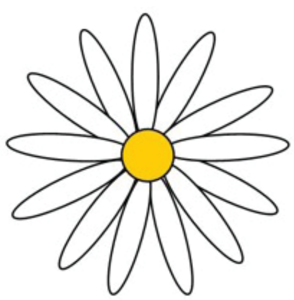


Fig. 1

Let's call the two players Alice and Bob; Alice gets the first move. If the rules allowed players to pick only one petal at a time, this game would be too easy! Since there is an odd number of petals, Alice would always win. However each player can choose either one or two, thus complicating matters. The truth is that in the game of Daisy, if Bob, the second player, knows what he is doing, Alice cannot win.

We'll examine the game from Bob's point of view. Alice makes the first move—we'll assume that she removed one petal (Fig. 2).

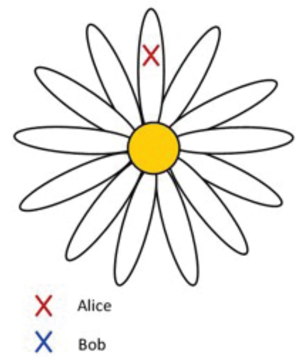


Fig. 2

Now Bob is looking at the remaining twelve petals, thinking, I have a chance to turn the tables in my favor! But which petals, and how many, should I pick? Well, if Bob picks one petal at a time, Alice will win no matter what. Thus, Bob wants to choose petals in such a way as to leave an even amount remaining. Easier said than done; since he can only pick two petals that are adjacent to each other, he might end up in a situation where there is an even number of petals on his turn, but all of them are not adjacent to any other petal. Consider Figure 3 below, which we will observe as an artbitrary but possible outcome of playing out the game. In this situation, Bob cannot pick two petals at once, since the rules dictate that the two petals must be adjacent.

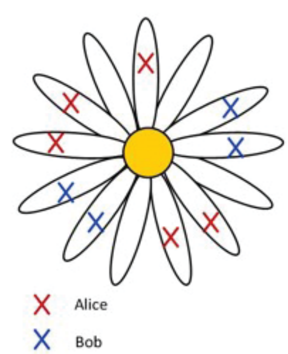


Fig. 3

In this case Bob would have no choice but to create an odd remainder of petals, beginning his spiral into doom. But he shouldn't despair! The truth is, as the second player, Bob has an advantage— he can control the consequences of Alice, the first player's actions. No matter what Alice does, Bob wants to be able to create an even number of petals on his turn and eventually force Alice to create an odd remainder. This is where the interesting properties of reflection symmetry come into play. Let me demonstrate.

Let's say Alice chooses one petal on her first move. Then let Bob choose two petals—the two petals directly opposite of Alice's (Fig. 4).

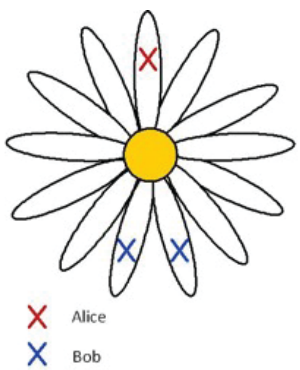


Fig. 4

By doing this, Bob has reconfigured the structure of the game. The daisy is now divided into two equal sections, with five available petals remaining in each section (Fig. 5).

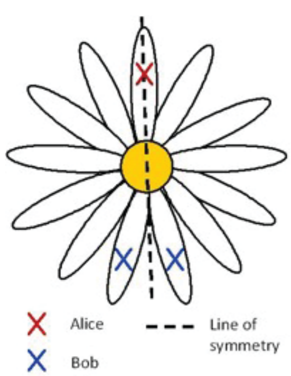


Fig. 5

Now that the game has a symmetrical configuration, what should Bob's next move be? If you think about it, a system in reflection symmetry allows him to do one thing: maintain the symmetry by mirroring any disturbances that are caused to the balance. These disturbances are, of course, the moves of the first player, Alice. Whatever Alice does, whether she chooses one or two petals, and whether she chooses them on the left or right side Bob can maintain the symmetry by mirroring her movements. That is, Bob will now choose the same number of petals directly opposite Alice's previous move. For example, Figure 6 below illustrates Alice's possible arbitrary selection of two petals, and Bob's subsequent mirroring of her movement.

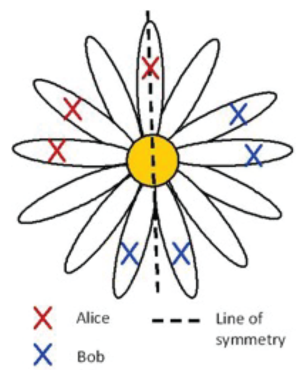


Fig. 6

Eventually, Alice will reach her turn and face a situation in which there are only two non-adjacent petals remaining (Fig. 7), or two non-adjacent pairs of petals, one pair on either side.

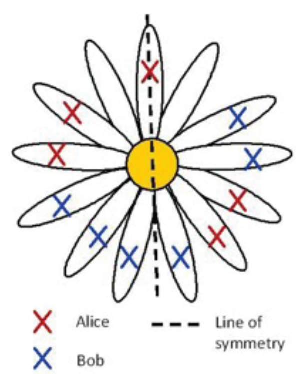


Fig. 7

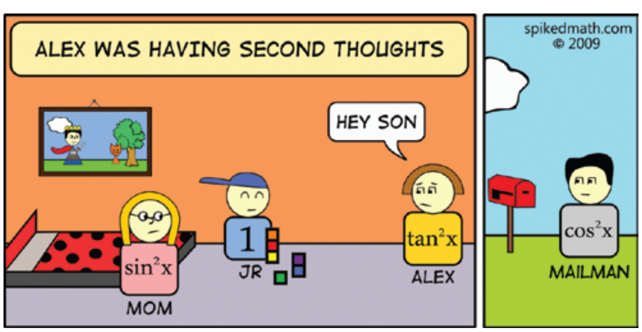
Since she can only remove one petal at a time in this case, Bob is guaranteed to remove the last petal.

Of course this is all based on the assumption that Alice, the first player, removes one petal first. But even if Alice removes two, the second player's win is still guaranteed. He just needs to create symmetry by removing the single petal opposite the ones Alice has chosen. The main goal is to create a flower with reflection symmetry; the order of petals removed at first doesn't matter. The rest is the same process.

So why does reflection symmetry work? Basically, the second player, Bob, wants to divide the daisy into two equal "sides", and he aims to force Alice to empty one of the sides first.

After Bob establishes the symmetry, Alice will be able to take petals from either the right or left side of the daisy. But by mimicking Alice's moves, Bob can still maintain the concept of two equal "sides". He maintains the balance, so to speak, and force Alice to disrupt the symmetry. The remainder will always be even, so eventually Alice will have no choice but to empty one of the sides. When Bob makes the symmetric move, he will automatically take the last petal and win the game.

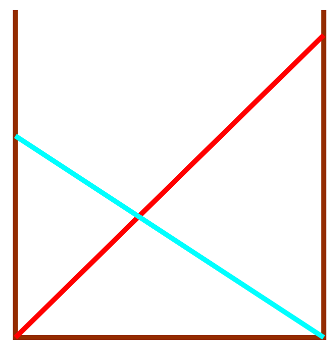
As a final thought, try to find a winning strategy for a 14-petal daisy!



Geometric Problems

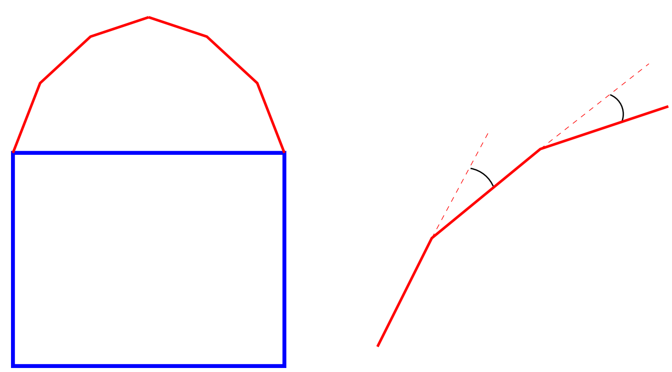
Here are two geometric problems submitted by readers.

This one is based on a suggestion of **Terence Coates.**



In a narrow alley a 35 foot ladder is placed up against one wall with its base touching the opposite wall. A 29 foot ladder is placed against the second wall with its base touching the first wall. The two ladders cross 11 2/3 feet above the ground (see the Figure). How wide is the alley?

And this one was suggested by **Gregory Akulov.**



The roof of a house is in flat parts with three different slopes: 7, m and 1 (the first three from left to right). If the the angle between each pair of slopes is the same (as shown in the figure), what is m?